Asymptotic Pion-Pion Elastic Scattering Amplitude*

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(Received 29 April 1963)

The elastic scattering amplitude for neutral pseudoscalar particles is calculated in the limit of high energies and small momentum transfer. We have kept all inelastic channels in the intermediate state and exchanging only two particles. This resulting expression gives the upper and lower bounds on the total cross section $b/s<\sigma(s)<a/s^{\epsilon}$, where $\epsilon>0$ and a, b are real constants. A special case of this elastic scattering amplitude is shown to be of the form suggested on the basis of the Regge theory of complex angular momentum. If the slope of the Regge trajectory is taken to be $a\simeq/50$ we obtain a total cross section which decreases very slowly $\sigma(s)=(k/16\pi)s^{-0.005}$, where k is some real constant.

I. INTRODUCTION

ONE of the most important features of strongly interacting particles at high energies is that the elastic scattering is almost completely concentrated in a forward cone. The total cross section appears to become smooth.[1] Although pion-pion interactions have not been observed at high energies, it is expected that they might also have similar characteristics.

A theoretical analysis, based on these considerations, has been performed for the pion-pion elastic scattering amplitude, using the Mandelstam representation of the scattering amplitude and taking into account all inelastic channels in the intermediate states. By keeping only the minimum number of particles exchanged, we obtain the asymptotic behavior of the amplitudes for small momentum transfer as the energy approaches infinity. This approach is based on the strip-approximation proposal of Domokos,[2] whereby one keeps only the asymptotic terms and then solves a Riemann-Hilbert boundary-value problem for the Mellin transform of the amplitude. The calculation will be carried out with the neglect of isospin. At high energies this model and the neglect of isospin appear to be experimentally suggested,[3] so that the present assumption may be quite realistic for pion-pion scattering.

The model developed here will allow us to obtain a general form of the scattering amplitude at large energies. From this one can obtain an upper and lower bound on the total cross section. We shall also show that a special case of the scattering amplitude has the same form as the one suggested by many authors on the basis of the Regge theory of complex angular momentum.[4]

In Sec. II, the integral equation for the scattering amplitude is derived keeping only the minimum number of particles exchanged. The asymptotic form of the integral equation will be given in Sec. III. We shall show that if no subtractions are needed in the dispersion relations the asymptotic integral equation has no solution. Using one subtraction and relating the subtraction constant to the total cross section, we obtain a general form of the elastic scattering amplitude. A special case of this amplitude is the one suggested on the basis of the Regge theory of complex angular momentum. Using the generalized form of the scattering amplitude it is shown that the total cross section must decrease as a power of the energy. Also, upper and lower bounds are given on the total cross section. Finally, using a special case of the amplitudes we obtain a cross section that is very nearly constant.

II. APPROXIMATE INTEGRAL EQUATION

Consider the following reactions illustrated by Fig. 1 and given by

$$\begin{aligned} &\mathrm{I} && \pi(P_1)+\pi(P_2)\to\pi(P_3)+\pi(P_4)\,,\\ &\mathrm{II} && \pi(P_1)+\pi(-P_4)\to\pi(P_3)+\pi(-P_2)\,,\\ &\mathrm{III} && \pi(P_1)+\pi(-P_3)\to\pi(-P_2)+\pi(P_4)\,,\end{aligned}$$

and let

$$\begin{aligned} &(h=c=m_\pi=1)\,,\\ &s=-(P_1+P_2)^2\,,\\ &t=-(P_1-P_4)^2\,,\\ &u=-(P_1-P_3)^2\,,\\ &4=s+t+u\,,\end{aligned}$$

where P_i is the four-momentum of the ith particle.

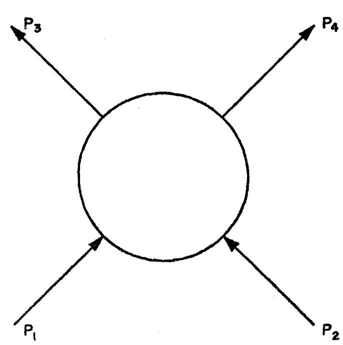

FIG. 1. Diagram describing the pion-pion elastic scattering amplitude.

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* Supported in part by the U. S. Atomic Energy Commission under contract A. T. (45-1)1388, program B.

[1] K. Winter, CERN Report 61-22 (unpublished).

[2] G. Domokos, Zh. Eksperim. i Teor. Fiz. 42, 538 (1962) [translation: Soviet Phys.—JETP 15, 377 (1962)].

[3] V. S. Barashenkov, Usp. Fiz. Nauk 72, 53 (1960) [translation: Soviet Phys.—Usp. 3, 689 (1961)].

[4] See, for example, B. M. Udgaonkar, Phys. Rev. Letters 8, 346 (1962).

The analytic properties of the amplitude may be written as follows[5]:

$$A(s,t,u)=\frac{1}{\pi^2}\int_4^\infty\int_4^\infty\frac{A_{st}(s',t')dt'ds'}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_4^\infty\int_4^\infty\frac{A_{su}(s',u')ds'du'}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int_4^\infty\int_4^\infty\frac{A_{ut}(u',t')du'dt'}{(u'-u)(t'-t)},\quad(1)$$

where the real spectral functions $A_{xy}(x,y)$ satisfy the relations

$$A_{st}=A_{us}=A_{tu}.\quad(2)$$

For convenience we will sometimes write $A(s,t,u)=A(s,t)$.

The absorptive part of the amplitude can be expressed as

$$A_x(x,y,z)=\frac{1}{\pi}\int_4^\infty\frac{A_{xy}(x,y')dy'}{y'-y}+\frac{1}{\pi}\int_4^\infty\frac{A_{xz}(x,z')dz'}{z'-z}.\quad(3)$$

We also have the following relations among the absorptive parts,

$$\begin{aligned}&A_x(x,y)=0\quad\text{if}\quad x<4,\\&A_s(s,t)=A_t(s,t)=A_u(s,t),\\&A_x(s,t)=A_x(s,4-s-t),\quad\text{where}\quad x=s,\,t,\text{ or }u.\end{aligned}\quad(4)$$

These equations only hold for no subtractions but can be generalized to the case of subtractions in the usual way.[5]

To obtain the integral equation for the spectral function keeping only two-particle intermediate states, we use the unitarity condition in channel III for $4<t<16$, inelastic processes in channel II being forbidden. The unitarity condition then has the form,[6]

$$A_u(z,t)=\frac{1}{4}\left(\frac{t-4}{t}\right)^{1/2}\int\int\frac{dz_1dz_2A(z_1,t)A^*(z_2,t)}{(1+2zz_1z_2-z_1^2-z_2^2-z^2)^{1/2}},\quad(5)$$

where $A_u(s,t)=\mathrm{Im}A(s,t)$ in channel III and $z_i=1+2s_i/(t-4)$. The range of the integral is over the region in which the radical is positive.

From the analytic properties of $A(s,t)$ and the fact that $\mathrm{Im}A_s=\mathrm{Im}A_u$, we obtain

$$\mathrm{Im}A_u(z,t)=\frac{1}{\pi}\left(\frac{t-4}{t}\right)^{1/2}\int\int\frac{dz_1dz_2\,A_s(z_1,t)A_s^*(z_2,t)}{(z^2-2zz_1z_2+z_1^2+z_2^2-1)^{1/2}},\quad(6)$$

[5] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).
[6] F. M. Kuni and I. A. Terentiv, Zh. Eksperim. i Teor. Fiz. **40**, 866 (1961) [translation: Soviet Phys.—JETP **13**, 607 (1961)].

where the range of the integral is again over the region in which the radical is positive.

Equation (6) is exact for $4<t<16$. We shall assume that it is valid for all t, i.e., that the contribution outside of the strip is small in Eq. (6) for large s.

III. ASYMPTOTIC SOLUTION

To find the asymptotic solution of Eq. (6) we will follow the same procedure as used by Domokos.[2]

Expanding Eq. (6) for large s, we obtain

$$\mathrm{Im}A_s(s,t)=\frac{1}{\pi}\left(\frac{t-4}{4}\right)^{1/2}\int_4^\infty\int_4^\infty\frac{ds_1ds_2\theta(x-1)A_s(s_1,t)A_s^*(s_2,t)}{(x^2-1)^{1/2}s_1s_2}+o(\mathrm{Im}A_s(s,t)),\quad(7)$$

where

$$x=\frac{z-z_1z_2}{[(z_2^2-1)(z_1^2-1)]^{1/2}}.$$

Application of the Mellin transformation

$$M_s^\mu(f(s))=\int_4^\infty f(s)s^{\mu-1}ds\quad(8)$$

to Eq. (7) yields the following:

$$M_s^\mu(\mathrm{Im}A_s(s,t))\simeq\frac{1}{\pi}\left(\frac{t-4}{t}\right)^{1/2}\frac{2^{2\mu-1}\Gamma(\frac{1}{2})\Gamma(1-\mu)}{\Gamma(\frac{3}{2}-\mu)(t-4)^\mu}|M_s^\mu(A_s(s,t))|^2\quad(9)$$

for μ real and $\mu<1$. $\Gamma(x)$ is a gamma function.

Here the following notation has been used. We define the notation as follows:

$$\phi(z)\simeq\psi(z)\quad\text{as}\quad z\to z_0,$$

if

$$\phi(z)=\psi(z)+o(\psi(z))\quad\text{as}\quad z\to z_0.$$

Consider the analytic properties of $M_s^\mu(A_s(s,t))$. We see that in the t plane for $\mathrm{Re}\mu<1$ it is an analytic function with the same analytic properties as $A_s(s,t)$. The left-hand cut of $A_s(s,t)$ gives a vanishing contribution for large s. Therefore, asymptotically in s we can write $M_s^\mu(A_s(s,t))$ in the following form:

$$M_s^\mu(A_s(s,t))\simeq\frac{t^n}{\pi}\int_4^\infty\frac{dt'\,\mathrm{Im}[M_s^\mu(A_s(s,t'))]}{t'^n(t'-t)}+P_{n-1}(t),\quad(10)$$

where $P_{n-1}(t)$ is an arbitrary polynomial of order $n-1$ in t. For real μ we have that

$$\mathrm{Im}M_s^\mu(A_s(s,t))=M_s^\mu(\mathrm{Im}A_s(s,t)).\quad(11)$$

Consider first the case where there are no subtractions and let

$$B(\mu,t)=\frac{1}{M_s{}^\mu(A_s(s,t))}. \quad (12)$$

Then from Eq. (9) for real μ and $\mu<1$, we obtain

$$\mathrm{Im}B(\mu,t)\simeq-\left(\frac{t-4}{t}\right)^{1/2}\frac{2^{2\mu-1}\Gamma(\frac{1}{2})\Gamma(1-\mu)}{(t-4)^\mu\pi\Gamma(\frac{3}{2}-\mu)}. \quad (13)$$

Thus $B(\mu,t)$ has the same analytic properties as $M_s{}^\mu(A_s(s,t))$ except that the poles of $M_s{}^\mu(A_s(s,t))$ will be the zeros of $B(\mu,t)$ and vice versa. The zeros of $M_s{}^\mu(A_s(s,t))$ are similar to the C.D.D. ambiguity[7] and will be assumed not to exist. Therefore, we have that

$$B(\mu,t)\simeq-2^{2\mu-1}\frac{\Gamma(\frac{1}{2})\Gamma(1-\mu)}{\pi\Gamma(\frac{3}{2}-\mu)}\times\int_4^\infty\left(\frac{t'-4}{t'}\right)^{1/2}\frac{dt'}{(t'-4)^\mu(t'-t)}, \quad (14)$$

and for $t\leq 0$ and $0<\mathrm{Re}\mu<1$

$$M_s{}^\mu(A_s(s,t))\simeq-\frac{\pi\sin(\pi\mu)}{2^{2\mu-1}(-t+4)^{\frac{1}{2}-\mu}\,{}_2F_1(\frac{1}{2},\frac{3}{2}-\mu;\frac{3}{2};\frac{1}{4}t)}. \quad (15)$$

Consider the case of $t=0$. Then

$$M_s{}^\mu(A_s(s,0))=-\pi\sin(\pi\mu), \quad \text{where} \quad 0<\mathrm{Re}\mu<1. \quad (16)$$

The inverse transformation of Eq. (16) does not exist. Since the total cross section is proportional to $A_s(s,0)$ this implies that the total cross section does not exist. This demonstrates that at least one subtraction is needed.

Now consider the case of one subtraction,

$$B(\mu,t)\simeq\frac{t}{\pi}\int_4^\infty\frac{\mathrm{Im}B(\mu,t')dt'}{t'(t'-t)}+f(\mu). \quad (17)$$

The subtraction term $f(\mu)$ is dependent on where the subtraction is made. If this is taken at $t=0$, we have a very simple interpretation of $f(\mu)$. In our notation $A(s,t)$ is normalized in such a way that

$$A_s(s,0)=\mathrm{Im}A(s,0)=s\sigma(s)/16\pi, \quad (18)$$

and, therefore,

$$\frac{1}{f(\mu)}\simeq\frac{1}{B(\mu,0)}=M_s{}^\mu(A_s(s,0))=\frac{1}{16\pi}\int_4^\infty s^\mu\sigma(s)ds. \quad (19)$$

Hence, $f(\mu)$ is simply related to the cross section.

From Eqs. (12), (13), and (17) we obtain

$$M_s{}^\mu(A_s(s,t))\simeq-\left\{\frac{2^{2\mu-1}\Gamma(\frac{1}{2})\Gamma(1-\mu)(t-4)^{\frac{1}{2}-\mu}}{\pi\Gamma(\frac{3}{2}-\mu)\pi t^{1/2}}\left[\cot\pi(1+\mu)-\frac{\Gamma(\mu)\Gamma(\frac{3}{2}-\mu)}{\pi\Gamma(\frac{3}{2})}+i\right]-f(\mu)\right\}^{-1}, \quad t>4$$

$$\simeq-\left[\frac{t2^{2\mu-1}\Gamma(\frac{1}{2})\Gamma(1-\mu)\Gamma(1+\mu)(4-t)^{\frac{1}{2}-\mu}\,{}_2F_1(\frac{3}{2},\frac{3}{2}-\mu;\frac{5}{2};\frac{1}{4}t)}{\pi^3\Gamma(\frac{3}{2})4^{3/2}}-f(\mu)\right]^{-1}, \quad t\leq 0 \quad (20)$$

where μ is real and $-1<\mathrm{Re}\mu<1$ for $t\neq 0$.

We now have an expression for the Mellin transform of the absorptive part of the elastic amplitude for real μ and can analytically continue for all μ.

First let us consider Eq. (20) for the case $t\leq 0$,

$$M_s{}^\mu(A_s(s,t))\simeq-\left[\frac{4^{\mu-1}\mu t(4-t)^{\frac{1}{2}-\mu}}{3\pi^2\sin(\pi\mu)}\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2}-\mu;\tfrac{5}{2};\tfrac{1}{4}t)-f(\mu)\right]^{-1}. \quad (21)$$

From Eq. (21) and with the aid of the convolution theorem for Mellin transformations[8] we can find the Mellin transform of $A(s,t)$.

From the fixed t dispersion relation and crossing symmetry we have that

$$A(s,t)=\frac{1}{\pi}\int_4^\infty\frac{ds'}{s'}A_s(s',t)\times\left[\frac{s}{s'-s}-\frac{s+t-4}{s'+s+t-4}\right]+P_0(t), \quad (22)$$

where $P_0(t)$ is the subtraction constant. For large s and fixed t, Eq. (22) may be expressed asymptotically as

$$A(s,t)\simeq\frac{s}{\pi}\int_4^\infty\frac{ds'}{s'}A_s(s',t)\left[\frac{1}{s'-s}-\frac{1}{s'+s}\right]. \quad (23)$$

The convolution theorem states that if for given functions $\Phi_1(z)$ and $\Phi_2(z)$ the Mellin transform, $\phi_1(s)$ and $\phi_2(s)$, exists and

$$\Phi(z)=\int_0^\infty\Phi_1(\rho)\Phi_2\left(\frac{z}{\rho}\right)\frac{d\rho}{\rho}, \quad (24)$$

then

$$\phi(s)=\phi_1(s)\phi_2(s),$$

[7] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[8] E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford University Press, New York, 1937), Chap. II.

where $\phi(s)$ is the Mellin transform of $\Phi(z)$. Application of this theorem to Eq. (23) gives

$$\pi M_{s,0}{}^{\mu}\left(\frac{A(s,t)}{s}\right)$$
$$=M_{s}{}^{\mu}\left(\frac{A_s(s,t)}{s}\right)\left[M_{s,0}{}^{\mu}\left(\frac{1}{1-s}\right)-M_{s,0}{}^{\mu}\left(\frac{1}{1+s}\right)\right], \quad (25)$$

where we have used the following notation

$$M_{s,0}{}^{\mu}\left(\frac{A(s,t)}{s}\right)=\int_0^\infty ds\, s^{\mu-1}\frac{A(s,t)}{s}. \quad (26)$$

The last term in Eq. (25) can easily be evaluated to give

$$M_{s,0}{}^{\mu}\left(\frac{1}{1-s}\right)-M_{s,0}{}^{\mu}\left(\frac{1}{1+s}\right)$$
$$=\frac{\pi}{\sin\pi\mu}[(-1)+(-1)^{\mu}], \quad (27)$$

for $0<\mathrm{Re}\mu<1$. With this result Eq. (25) becomes

$$M_{s,0}{}^{\mu}\left(\frac{A(s,t)}{s}\right)\simeq M_{s}{}^{\mu}\left(\frac{A_s(s,t)}{s}\right)\left[\frac{(-1)+(-1)^{\mu}}{\sin\pi\mu}\right], \quad (28)$$

and by means of Eq. (21) we obtain

$$M_{s,0}{}^{\mu}(A(s,t))\simeq\frac{-[1+(-1)^{\mu}]}{[4^{\mu-1}\mu t(4-t)^{\frac{1}{2}-\mu}/3\pi^2]\,{}_2F_1(\frac{3}{2},\frac{3}{2}-\mu;\frac{5}{2};\frac{1}{4}t)-f(\mu)\sin\pi\mu}. \quad (29)$$

Since the problem had no solution for zero subtractions we have the restriction on μ: $-1<\mathrm{Re}\mu\leq 0$ [cf. Eq. (16)]. From this restriction, we can obtain an upper and lower bound on the amplitude. We have

$$b\leq|A(s,t)|\leq as, \quad (30)$$

where a and b are some real constants and $t\leq 0$ and s is very large. This condition gives us the following bounds on the total cross section:

$$a's^{-1}\leq\sigma(s)\leq b', \quad (31)$$

where a' and b' are some real constants.

The most general form of the asymptotic behavior of the total cross section which satisfied the inequality of Eq. (31) is

$$\sigma(s)\simeq ks^{-\beta-1}(\ln s)^{\alpha}L(\ln s), \quad (32)$$

where $0\leq-\beta\leq 1$ and k is a real constant. $L(\ln s)$ is called a "langsam wachsende Funktion" and has the following properties[9]:

$$\text{(i)}\quad \lim_{s\to\infty}(\ln s)^{\epsilon}L(\ln s)=0 \quad\text{for any } \epsilon>0,$$
$$\text{(ii)}\quad \lim_{s\to\infty}(\ln s)^{-\epsilon}L(\ln s)=0 \quad\text{for any } \epsilon>0, \quad (33)$$
$$\text{(iii)}\quad \lim_{s\to\infty}\frac{L(a\ln s)}{L(\ln s)}=1 \quad\text{for any } a>0.$$

To obtain the most general form of the subtraction constant we must take the Mellin transform of Eq. (32). To carry this out we will use the following theorem[10]:

Theorem. If $F(\ln t)$ is a J function[10] with the asymptotic property

$$F(t)\simeq A(\ln t)^{\alpha}L(\ln t) \quad\text{for}\quad t\to\infty,$$

where A is any arbitrary constant, $\mathrm{Re}\alpha>-1$, and for $t\geq t_0>0$ the continuous, positive function $L(t)$ satisfies the condition

$$\frac{L(a\ln t)}{L(t)}\to 1 \quad\text{for}\quad t\to\infty \quad\text{and}\quad a>0,$$

then there exists $M_{t,0}{}^{s}(F(t))=f(s)$ for $\mathrm{Re}s>0$. The function $f(s)$ is singular at $s=0$ and can be asymptotically represented as

$$f(s)\simeq\frac{A\Gamma(\alpha+1)}{s^{\alpha+1}}L\left(\frac{1}{s}\right) \quad\text{for}\quad \mathrm{Re}s\to 0.$$

This theorem is only valid for values of $\alpha\leq-1$; however, since we will not need an explicit expression for the case of $\alpha\leq-1$ we will exclude it here.

Using this theorem and Eqs. (19) and (32) gives the subtraction constant

$$f(\mu)^{-1}\simeq\frac{k}{16\pi}\frac{\Gamma(\alpha+1)}{(\beta-\mu)^{\alpha+1}}L\left(\frac{1}{\beta-\mu}\right), \quad (34)$$

for $\alpha>-1$.

Equation (29) now becomes

$$M_{s,0}{}^{\mu}(A(s,t))\simeq\frac{-[1+(-1)^{\mu}]}{[4^{\mu-1}\mu t(4-t)^{\frac{1}{2}-\mu}/3\pi^2]\,{}_2F_1(\frac{3}{2},\frac{3}{2}-\mu;\frac{5}{2};\frac{1}{4}t)-[16\pi(\beta-\mu)^{\alpha+1}\sin(\pi\mu)/k\Gamma(\alpha+1)L(1/\beta-\mu)]}, \quad (35)$$

for $\alpha>-1$.

[9] See, for example, J. Karamata, Mathematica (Cluj, Rumänien) 4, 45 (1930), or G. Doetsch, *Theorie und Anwendung der Laplace-Transformation* (Julius Springer Verlag, Berlin, 1937).

[10] G. Doetsch, *Handbuch der Laplace-Transformation* (Verlag Berkhauser, Basel, 1950), Band I, Theorem 7, p. 460.

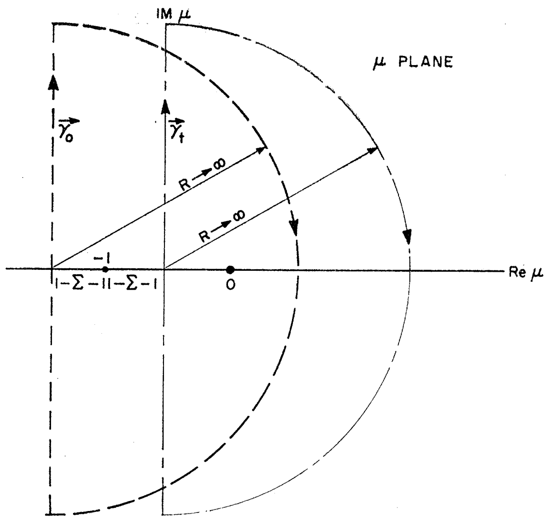

FIG. 2. Contours in the μ plane for the inverse Mellin transformation.

We will now show that the total cross section will asymptotically go to zero at least as fast as a power of the energy, i.e.,

$$\sigma(s)=O(s^{-\epsilon}), \quad \text{where} \quad \epsilon>0. \tag{36}$$

Therefore, a logrithmic decrease or anything slower is forbidden.

To show this we will use Eqs. (20a), (21), and (34). First consider the special case $A_s(s,0)\simeq(ks/16)$ [i.e., $\sigma(s)_{s\to\infty}\simeq k$]. For this example we have $\alpha=0$, $\beta=-1$, $L(1/(\beta-\mu))=1$.

Equation (21) becomes

$$M_s{}^{\mu}(A_s(s,0))=\frac{k}{16\pi}\frac{1}{(-1-\mu)}, \tag{37}$$

where $\mu<-1$, and

$$M_s{}^{\mu}(A_s(s,t))=\frac{-\sin(\pi\mu)(k/16\pi)}{(k\mu t/96\pi^2)+(1+\mu)\sin\pi\mu}, \tag{38}$$

for t sufficiently small and $-1<\mu\leq 0$.

The conditions on μ in Eqs. (37) and (38) mean that the path of integration for the inverse transformation must cross the real axis where the inequalities are satisfied, i.e., the path of integration must cross real μ axis somewhere between $-1<\mu\leq 0$ for $t<0$ and $\mu<-1$ for $t=0$ (see Fig. 2), where ϵ and ϵ' are greater than zero.

The absorptive part of the amplitude is

$$A_s(s,t)=\frac{1}{2\pi i}\int_{\gamma} M_s{}^{\mu}(A_s(s,t))s^{-\mu}d\mu, \tag{39}$$

where

$$\gamma=\gamma_0 \quad \text{for} \quad t=0$$
$$=\gamma_t \quad \text{for} \quad t<0.$$

The only way that the absorptive amplitude can exist and satisfy these conditions is for the contour γ_0 to be homotopic to γ_t relative to the μ plane for $t<0$. In order to see if this is true we must look at Eqs. (37) and (38) to see how the singularities behave as t becomes finite.

Equation (37) shows that we have a simple pole at $\mu=-1$ for $t=0$.

Equation (38) shows that for $t<0$ the pole about $\mu=-1$ splits into two poles, one moving on the left and the other to the right as shown in Fig. 3.

Therefore, γ_0 cannot be homotopic to γ_t relative to μ for $t<0$ and hence, the conditions are not satisfied. The same argument still holds for $\alpha=1, 2\cdots n$. For $\alpha>-1$ and a noninteger we have a branch point which does not move with t, so γ_0 is still not homotopic to γ_t. Also if $L(-1/(\mu+1))\neq 1$ it will introduce an additional singularity at $\mu=-1$ which will again hinder this condition [cf. Eq. (34)]. To study the type of singularities for $\alpha\leq -1$ in Eq. (21) we make use of the following identity.

Let $\Phi(z)$ be a function whose Mellin transform is $\phi_1(s)$; then the Mellin transform $\phi_2(s)$ of $(\ln z)^{-n}\Phi(z)$ is

$$\phi_2(s)=\int_{-\infty}^{s}\int_{-\infty}^{s_1}\cdots\int^{s_{n-1}}\phi_1(s_n)ds_n\cdots ds_1, \tag{40}$$

or

$$d^n\phi_2(s)/ds^n=\phi_1(s). \tag{41}$$

For $\alpha\leq -1$ and n sufficiently large (i.e., $n>|\alpha|-1$), we have that the nth derivative of the subtraction constant is of the form

$$\frac{d^n}{d\mu^n}\left(\frac{1}{f(\mu)}\right)=\frac{k}{16\pi}\frac{\Gamma(\lambda+1)}{(\beta-\mu)^{\lambda+1}}L\left(\frac{1}{\beta-\mu}\right), \tag{42}$$

where $\lambda>-1$. From Eq. (42) we see that for the case $\alpha\leq -1$ we again have the same type of singularities as

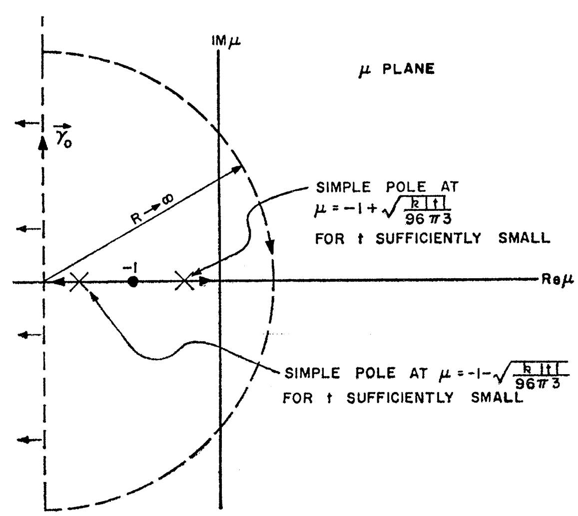

FIG. 3. Position of singularities in the μ plane for $t\neq 0$.

for $\alpha > -1$ in Eq. (21) and the same arguments as given above prevent γ_0 from being homotopic to γ_t relative to the μ plane for $t<0$ and $\alpha \leq -1$.

We have now shown that

$$b/s \leq \sigma(s) \leq a s^{-\epsilon}, \tag{43}$$

where $\epsilon > 0$ and a, b are some real constants.

Keeping in mind the above inequalities we are ready to take the inverse transformation of $M_{s,0}{}^{\mu}(A(s,t))$. If we take Eq. (35) and keep only the lowest order terms for t sufficiently small we have

$$A(s,t) \simeq -\frac{[1+(-1)^{\beta}]}{\sin(\pi\beta)} \frac{k}{16\pi} s^{-\beta}(\ln s)^{\alpha} L(\ln s) \times\left[1+\frac{t\beta\Gamma(\alpha+1)k(\ln s)^{2\alpha+1}}{96\pi^3 \sin(\pi\beta)} L^2(\ln s)+O(t^2)\right], \tag{44}$$

where $0 \leq -\beta < 1$ and $\alpha > -1$.

Notice that the Regge behavior is a special case of Eq. (35). If we assume, as in the Regge case, that the partial-wave amplitude is meromorphic in the l plane with only simple poles, this is equivalent to our case of having only simple poles in the μ plane. In this special case we get the amplitude

$$A(s,t) \simeq -\frac{k}{16\pi} \frac{s^{-\beta+at}}{\sin \pi\beta} [1+(-1)^{\beta}], \tag{45}$$

where

$$a = \frac{\beta k}{\sin(\pi\beta) 96\pi^3}, \tag{46}$$

and

$$0 \leq \beta < +1. \tag{47}$$

We see from Eq. (46) that the slope of the Regge trajectory a for small t is strongly dependent on the asymptotic power of the total cross section. Hence, if we know the slope of the trajectory we know the asymptotic cross section or vice versa. If we take the case of $a \sim 1/50$, this implies that $\beta \sim -0.995$ giving

$$A(s,t) \simeq \frac{k s^{0.995+t/50}[1+(-1)^{-0.995}]}{16\pi \sin(\pi 0.995)}, \tag{48}$$

and

$$\sigma(s) = (k/16\pi) s^{-0.005}. \tag{49}$$

The relation between the trajectory a and the asymptotic power β seems to give surprisingly excellent agreement with experiment.[11]

IV. CONCLUSION

In principle, the spectral function can also be obtained from Eq. (20) for $t \geq 4$; however, the arguments will be the same as above and no new restrictions on the amplitudes will arise that have not already appeared for the case of $t \leq 0$. However, it is interesting to note that whenever the subtraction constant $f(\mu)$ is meromorphic in the μ plane with only simple poles [i.e., $\sigma(s) \simeq (k/16\pi)s^{\beta}$], we will get a spectral function of the form

$$A_s(s,t) \simeq s^{\alpha(t)}, \tag{50}$$

where $\alpha(t)$ is a complex function of t. This form of the spectral function is just the one suggested by many authors on the basis of the Regge theory of complex angular momentum.

The general form of the amplitude in Eq. (35) easily lends itself to a study of the general properties of the amplitude. From this amplitude we have obtained in Eq. (43) an upper and lower bound on the total cross section,

$$b/s \leq \sigma(s) \leq a/s^{\epsilon},$$

where $\epsilon > 0$ and a, b are some real constants.

With the additional assumption that $f(\mu)$ has only simple poles, Eqs. (48) and (49) give remarkable agreement with experiments. This amplitude will give a total cross section which is very nearly constant and the elastic cross section will be almost completely concentrated in a forward cone.

In this same elastic approximation to the unitarity condition, the amplitude for the pion-nucleon scattering is given in terms of the pion-pion amplitude and can now likewise be solved in terms of it. Finally the nucleon-nucleon scattering amplitude is given in terms of the pion-nucleon amplitude.

ACKNOWLEDGMENTS

I would like to thank Professor E. M. Henley for his interest and patient criticism of this study. I also thank Dr. Graham Frye, Dr. Marshall Baker, and Dr. Ivan Muzinich for their helpful discussions.

[11] See, for example, W. F. Baker, E. W. Jenkins, and R. L. Reed, Phys. Rev. Letters **9**, 221 (1962) and Ref. 1.